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Robustness Study for a Linear Growth Model

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For the linear growth curve model introduced by Potthoff and Roy (*Biometrika* **51** (1964), 313–326), various likelihood ratio tests and some ad hoc tests are available for the location and scale parameters on the basis of normally distributed error components. We study these tests under the assumption of elliptical (or spherical) distributions of the error components and show that these tests are null robust; and the tests for the location parameters are shown to be unbiased. These results are extended to the linear growth model in complex variables having elliptical (or spherical) complex distributions. © 1988 Academic Press, Inc.

1. INTRODUCTION

A general linear model introduced by Potthoff and Roy [11] is given in some extended form as

$$Y = B\xi A + \theta\eta V, \tag{1.1}$$

where Y is a $p \times n$ observation matrix on n individuals, A and B are known matrices of respective orders $m \times n$ and $p \times q$ and $\varepsilon = \theta\eta V$ is $p \times n$ error matrix. Here, ξ is a $q \times m$ matrix of location parameters, θ is a $p \times p$ non-singular matrix of scale parameters, V is an $s \times n$ known matrix of rank s $(\leqslant n)$, and η is a $p \times s$ random matrix. In the usual Potthoff and Roy's model [11], $V = I_n$ and the elements of η are independently distributed as $N(0, 1)$. For the robustness study, we shall assume that the density function of η exists and it is given by

$$f_0(\eta) \qquad \text{for all} \quad \eta \in R^{p \times s}, \tag{1.2}$$

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where f_0 does not contain θ and ξ as parameters and $R^{p\times s}$ is a set of all $p\times s$ matrices defined on the real field R. Let $O(m)$ denotes a set of $m\times m$ orthogonal matrices. Then, we shall say that the distribution of η is spherical if and only if

$$f_0(\eta)=f_0(\Delta_1\eta\,\Delta_2) \qquad \text{for all} \quad \Delta_1\in O(p) \text{ and } \Delta_2\in O(s). \tag{1.3}$$

If this condition (1.3) is satisfied, then it is easy to see that the density function of η must be a symmetric function of the nonzero eigenvalues of $\eta\eta'$ (or $\eta'\eta$); see, for example, David [3]. We shall assume that $s>p$ and when $p=1$, the density function of η is $f(\eta\eta')$. If the moments of η exists, then it is easy to see that

$$E\eta=0, \qquad E\eta_{ij}^2=c(>0), \qquad \text{and} \qquad E\eta_{ij}\eta_{i'j'}=0 \tag{1.4}$$

for all $i\neq i'$ and/or $j\neq j'$, i, $i'=1, 2, ..., p$ and j, $j'=1, 2, ..., s$ with η_{ij} being the (i, j) the element of η. From (1.4), we have

$$E(\eta T\eta')=c(\operatorname{tr} T)I_p \qquad \text{for any symmetric matrix } T. \tag{1.5}$$

Now, we shall indicate some of the recent work on the model (1.1). Kariya [4] and Kariya and Sinha [5] considered the canonical model with $V=I_n$ and A and B full rank matrices. Under the canonical model, they obtained locally best invariant tests for the location parameters and for $B=I_p$, the robustness study is given for the location parameters when the density function of η is $g(\operatorname{tr}\eta\eta')$, while Chmielewski [2] and Kariya and Sinha [5] study some property of the sphericity test due to $H_0(\theta\theta'=\sigma^2I)$ when $B=I_p$, $V=I_n$, and the density of η is $g(\operatorname{tr}\eta\eta')$. Further, Anderson and Fang [1] and Krishnaiah *et al.* [9, 10] have developed some results on the canonical correlation matrices and regression matrices when the density of η is $g(\operatorname{tr}\eta\eta')$, $B=I_p$, and $V=I_n$. We study some of these problems when $q\leqslant p$ and $s\leqslant n$. Further, we shall assume the density of η under condition (1.3) for the tests for location parameters, while for other problems we shall assume the density of η as $g(\operatorname{tr}\eta\eta')$. Khatri [8] has obtained some results for the location parameters, but some of them will be included as a ready reference.

We extend the above results to a general linear model in complex random variables; namely in (1.1) model, Y, B, A, and V are known matrices of respective orders $p\times n$, $m\times n$, $p\times q$, and $s\times n$ defined on the complex field $\mathscr{C}$, $\xi\in\mathscr{C}^{q\times m}$ and $\theta\in\mathscr{C}^{p\times p}$ are unknown parameters such that θ is nonsingular, and η is a complex random matrix of order $p\times s$ $(s>p)$ whose density function is $f_0(\eta)$ $\{=f_0(\Delta_1\eta\,\Delta_2)$ for all unitary matrices Δ_1 and Δ_2 of respective orders $p\times p$ and $s\times s\}$, a symmetric function in the eigenvalues of $\eta\eta^*$. Here, η^* denotes the conjugate transpose of η. As in (1.5), we have

$$E\eta=0$$

and

$$E(\eta T \eta^*) = c(\text{tr } T) I_p \text{ for any Hermitian matrix } T. \tag{1.6}$$

In model (1.1), reparametrizing the location parameters ξ, we shall assume throughout the paper that Rank $B = q$ and Rank $A = m$. Let A_1 be an $(n-m) \times n$ matrix such that $A_1 A^* = 0$ and $A_1 A_1^* = I_{n-m}$. Let us denote

$$\begin{aligned} S &= Y A_1^* (A_1 V^* V A_1^*)^- A_1 Y^* \\ &= Y[V_0^- - V_0^- A^* (A V_0^- A^*)^{-1} A V_0^-] Y^*, \end{aligned} \tag{1.7}$$

where $V_0 = V^*V + A^*CA$, V_0^- is a g-inverse of V_0, C is an arbitrary $m \times m$ positive definite matrix, and $\text{Rank}(V_0) = \text{Rank}(V^*, A^*) = s_1$ (say). Using (1.6),

$$ES = [\text{tr}\{V A_1^* (A_1 V^* V A_1^*)^- A_1 V^*\}] \Sigma c = c(s-t)\Sigma, \tag{1.8}$$

where $\Sigma = \theta\theta^*$, $s - t = \text{Rank}(A_1 V^*)$, and $t = \text{Rank}(A V_0^- V^*)$.

Now, if Σ is known, then the generalized least squares estimates of ξ are those which minimize

$$\phi_j = \text{tr}_j \eta\eta^* = \text{tr}_j\{\Sigma^{-1}(Y - B\xi A)(V^*V)^- (Y - B\xi A)^*\} \tag{1.9}$$

subject to $(Y - B\xi A)(I - (V^*V)^- V^*V) = 0$, where $V(V^*V)^- V^* = I_s$ for any g-inverse $(V^*V)^-$, and $\text{tr}_j P =$ sum of all the principal minors of order j from P. When $j = 1$, we get the usual least squares estimates and when $j = p$, we get the different estimate of ξ. This was done by Srivastava and Khatri [12] and arrived at the maximum likelihood estimates of ξ for the normal variates. For a general j, the answer from (1.9) is not explicit and hence, we shall consider the situation when Σ is unknown. We shall obtain the generalized least squares estimates of ξ as those values which minimise

$$\phi_{1j} = \text{tr}_j\{S^{-1}(Y - B\xi A)(V^*V)^-(Y - B\xi A)^*\} \tag{1.10}$$

subject to $(Y - B\xi A)(I - (V^*V)^- V^*V) = 0$. This is done in Section 2. Section 3 establishes an important property of BLUE (or best linear unbiased estimate) of ξ when Σ is known. Section 4 deals with the robustness property and unbiasedness of the tests for location parameters when the distribution of η is spherical (as defined above). Sections 5 and 6 introduce the concept of null robustness, which means that the null distributions of the test procedures will not depend on the structure of g. In these sections, the density function of η is $g(\text{tr } \eta\eta^*)$ and the test procedures are considered for the hypothesis $\Sigma = \theta\theta^* = \sigma^2 G$, G being known or $\Sigma = \sigma_1 G + \sigma_2 \mathbf{w}\mathbf{w}^*$, G and $\mathbf{w} = B\boldsymbol{\delta}$ for some $\boldsymbol{\delta}$ being known, or $\Sigma = \text{diag}(\Sigma_{11}, \Sigma_{22})$ under some restriction on B. All the tests developed for normal distributions are shown to be null-robust.

2. Generalized Least Squares Estimates of ξ

In (1.7), we introduced A_1, V_0, and V_0^- matrices. The following properties are easy to establish:

$$(I - V_0 V_0^-)(V^*, A^*) = 0 \qquad \text{for all} \quad V_0^-$$

and

$$A V_0^- A^* = A V_0^+ A^* \qquad \text{is nonsingular,} \tag{2.1}$$

where V_0^+ denotes the unique Moore–Penrose inverse of V_0. Note that $C_1 = (V_0^- A^*, A_1^*)$ is nonsingular on account of $\binom{A}{A_1}(V_0^- A^*, A_1^*)$ being nonsingular.

Since $A_1 V^* V V_0^- A^* = A_1 V_0 V_0^- A^* = A_1 A^* = 0$,

$$s = \text{Rank } V = \text{Rank}(V V_0^- A^*, V A_1^*) = \text{Rank}(V V_0^- A^*) + \text{Rank}(V A_1^*),$$

which proves the result of (1.8). Let $V_1 = V C_1$ and $V_{01} = V_0 C_1$. Then Rank $V_1 = s$ and, for any g-inverse $(V_1^* V_1)^-$ of $V_1^* V_1$, we must have

$$V_1 (V_1^* V_1)^- V_1^* = I_s = V_1 (V_1^* V_1)^+ V_1^* = V(G + G_1) V^*, \tag{2.2}$$

where $G = V_0^- A^* (A V_0^- V^* V V_0^- A^*)^+ A (V_0^-)^*$ and $G_1 = A_1^* (A_1 V^* V A_1^*)^+ A_1$. Thus, $G + G_1$ is a g-inverse of $V^* V$. Using this in the restrictions $Y(I - V_0^- V_0) = 0$ and $(Y - B\xi A)[V_0^- V_0 - (G + G_1) V^* V] = 0$, we can write them as

$$Y = Y V_0^- V_0 \qquad \text{and} \qquad (Z - B\xi) A = (Z - B\xi) A G V^* V, \tag{2.3}$$

where $Z = Y V_0^- A^* (A V_0^- A^*)^{-1}$. Similarly, we can write

$$(Y - B\xi A)(V^* V)^- (Y - B\xi A)^* = S + (Z - B\xi) A G A^* (Z - B\xi)^*. \tag{2.4}$$

Observe that $(I - G V^* V) G = 0$ and the restrictions (2.3) will be utilized after minimizing ϕ (or ϕ_{1j}) with respect to ξ without any restrictions. Hence, the problem reduces to minimize

$$\phi_{1j} = \text{tr}_j[I_p + S^{-1}(Z - B\xi) A G A (Z - B\xi)^*] \tag{2.5}$$

with respect to ξ. Note that defining $\text{tr}_0 R = 1$,

$$\begin{aligned}
\phi_{1j} &= \sum_{i=0}^{j} \binom{p-i}{j-i} \text{tr}_i \{S^{-1}(Z - B\xi) A G A (Z - B\xi)^*\} \\
&= \sum_{i=0}^{j} \binom{p-i}{j-i} \text{tr}_i \{F^* (Z - B\xi)^* S^{-1} (Z - B\xi) F\} \quad \text{with } A G A^* = F F^* \\
&= \sum_{i=0}^{j} \binom{p-i}{j-i} \text{tr}_i \{F^* Z^* (S^{-1} - S^{-1} B (B^* S^{-1} B)^{-1} B^* S^{-1}) Z F \\
&\quad + F^* (Z_1 - \xi)^* (B^* S^{-1} B)(Z_1 - \xi) F\},
\end{aligned}$$

where

$$Z_1=(B^*S^{-1}B)^{-1}B^*S^{-1}Z=(B^*S^{-1}B)^{-1}B^*S^{-1}YV_0^-A^*(AV_0^-A^*)^{-1}$$

is the maximum likelihood estimate of ξ under normal distribution of η. Now,

$$\phi_{1j}\geqslant\sum_{i=0}^{j}\binom{p-i}{j-i}\operatorname{tr}_i\{F^*Z^*(S^{-1}-S^{-1}B(B^*S^{-1}B)^{-1}B^*S^{-1})ZF\}$$

and the equality will hold if and only if

$$F^*(Z_1-\xi)^*(B^*S^{-1}B)(Z_1-\xi)F=0 \qquad \text{or} \qquad (Z_1-\xi)AG=0.$$

Now using condition (2.3), we see that

$$\operatorname{Min}\phi_{1j}=\operatorname{tr}_j(S^{-1}\hat{\Sigma}) \qquad \text{at} \quad \hat{\xi}=Z_1, \tag{2.6}$$

where

$$\hat{\Sigma}=S+(I-B(B^*S^{-1}B)^{-1}B^*S^{-1})WW^*(I-S^{-1}B(B^*S^{-1}B)^{-1}B^*)$$

and $W=ZF$. Notice that $\hat{\Sigma}$, except for a constant multiplier, is the maximum likelihood estimate of Σ under the normality of the variables. A similar result can be established for

$$[\operatorname{tr}_j S\{(Y-B\xi A)(V^*V)^-(Y-B\xi A)^*\}^{-1}]^{-1}.$$

Hence, we get

THEOREM 2.1. (a) *The generalized least squares estimate of* ξ *is* Z_1, *obtained by minimizing* $\operatorname{tr}_j\{S^{-1}(Y-B\xi A)(V^*V)^-(Y-B\xi A)^*\}$ *or* $\{\operatorname{tr}_j S[(Y-B\xi A)(V^*V)^-(Y-B\xi A)^*]^{-1}\}^{-1}$ *subject to* $(Y-B\xi A)(I-(V^*V)^-V^*V)=0$.

(b) *When* Σ *is known, then BLUE for* ξ *is* $Z_{(1)}=(B^*\Sigma^{-1}B)^{-1}B^*\Sigma^{-1}YV_0^-A^*(AV_0^-A^*)^{-1}$ *and it is obtained by minimizing*

$$\operatorname{tr}_j\{\Sigma^{-1}(Y-B\xi A)(V^*V)^-(Y-B\xi A)^*\}$$

subject to the restrictions $(Y-B\xi A)=(Y-B\xi A)(V^*V)^-(V^*V)$.

3. A PROPERTY OF BLUE WHEN Σ IS KNOWN

Let the BLUE for $C\xi D$ be G_1YG_2 (or the BLUE for $(D'\otimes C)\operatorname{vec}\xi$ be $(G_2'\otimes G_1)\operatorname{vec}Y$, where $P\otimes Q$ denotes the Kronecker product of P with Q

defined as $P \otimes Q = (p_{ij}Q)$ with $P = (p_{ij})$ and $(\text{vec } Y)' = (\mathbf{y}_1', \mathbf{y}_2', ..., \mathbf{y}_n')$ with $\mathbf{y}_i$ being the ith column vector of Y), and let any unbiased estimate of $C\xi D$ be $L_1 Y L_2$. Now, using property (1.3), it is easy to see that

$$E(\text{vec } \eta)(\text{vec } \eta)^* = cI_{ps} \text{ for some constant } c \text{ depending on } f_0.$$

Hence, on account of the BLUE property,

$$V[(L_2' \otimes L_1) \text{ vec } Y] - V[(G_2' \otimes G_1) \text{ vec } Y]$$

is positive semi-definite or

$$c((L_2^* V^* V L_2)' \otimes L_1 \theta\theta^* L_1^*) - c((G_2^* V^* V G_2)' \otimes G_1 \theta\theta^* G_1^*)$$

is positive semi-definite. This is true if and only if $L_2^* V^* V L_2 - G_2^* V^* V G_2$ and $L_1 \Sigma L_1^* - G_1 \Sigma G_1^*$ are positive semi-definite. Let M_1 and M_2 be any positive definite matrices and define

$$N_1 = M_1(G_1 Y G_2 - C\xi D)\, M_2(G_1 Y G_2 - C\xi D)^*$$
$$N_2 = M_1(L_1 Y L_2 - C\xi D)\, M_2(L_1 Y L_2 - C\xi D)^*.$$

Let $h_i = g(\lambda_{1i}, \lambda_{2i}, ..., \lambda_{ci})$ for $i = 1, 2$, where g is a nonnegative increasing function of each λ_{ji} $(j = 1, 2, ..., c)$ separately and $\lambda_{1i} \geqslant \lambda_{2i} \geqslant \cdots \geqslant \lambda_{ci} \geqslant 0$ are the nonzero eigenvalues of N_i $(i = 1, 2)$. Then, we shall show that

$$P(h_1 \leqslant d) \geqslant P(h_2 \leqslant d) \qquad \text{for all} \quad d \geqslant 0, \tag{3.1}$$

provided the density of η is $f(\eta\eta^*)$, a symmetric function of the eigenvalues of $\eta\eta^*$. To prove (3.1), notice that

$$N_1 = M_1 G_1 \theta\eta(VG_2 M_2 G_2^* V^*)\, \eta^* \theta^* G_1^*$$
$$N_2 = M_1 L_1 \theta\eta(VL_2 M_2 L_2^* V^*)\, \eta^* \theta^* L_1^*.$$

Since h_1 is a function of the nonzero eigenvalues of N_1, h_1 is a function of the nonzero eigenvalues of

$$(\theta^* G_1^* M_1 G_1 \theta)\, \eta(VG_2 M_2 G_2^* V^*)\eta^*.$$

Let us denote $(\theta^* G_1^* M_1 G_1 \theta) = \Gamma D_{\alpha(1)} \Gamma^*$, $\theta^* L_1^* M_1 L_1 \theta = \Gamma_1 D_{\alpha(2)} \Gamma_1^*$, $VG_2 M_2 G_2^* V^* = \Delta D_{\beta(1)} \Delta^*$, and $VL_2 M_2 L_2^* V^* = \Delta_1 D_{\beta(2)} \Delta_1^*$, where Γ, Γ_1, Δ, and Δ_1 are unitary matrices, $D_{\alpha(i)} = \text{diag}(\alpha_1(i), \alpha_2(i), ..., \alpha_p(i))$, $\alpha_1(i) \geqslant \alpha_2(i) \geqslant \cdots \geqslant \alpha_p(i) \geqslant 0$ and $D_{\beta(i)} = \text{diag}(\beta_1(i), ..., \beta_s(i))$ with $\beta_1(i) \geqslant \beta_2(i) \geqslant \cdots \geqslant \beta_s(i) \geqslant 0$ for $i = 1, 2$. Notice that $\alpha_j(1)$'s are the nonzero eigenvalues of $M_1 G_1 \Sigma G_1^*$ while $\alpha_j(2)$'s are the nonzero eigenvalues of $M_1 L_1 \Sigma L_1^*$. Since $M_1^{1/2}(L_1 \Sigma L_1^*) M_1^{1/2} - M_1^{1/2}(G_1 \Sigma G_1^*) M_1^{1/2}$ is positive semi-definite matrix, so $\alpha_j(2) \geqslant \alpha_j(1)$ for all $j = 1, 2, ..., p$. Similarly $\beta_j(2) \geqslant \beta_j(1)$ for $j = 1, 2, ..., s$. Further,

$$f_0(\eta) = f_0(\Gamma^* \eta \, \Delta) = f_0(\Gamma_1^* \eta \, \Delta_1)$$

and hence

$$P(h_i \leqslant d) = P(g(\lambda_{1i}, \lambda_{2i}, ..., \lambda_{ci}) \leqslant d)$$

for $i = 1, 2$, where λ_{ji} $(j = 1, 2, ..., c)$ are the nonzero eigenvalues of $(D_{\alpha(i)} \eta \, D_{\beta(i)} \eta^*)$ for $i = 1, 2$. Defining

$$D^{1/2}_{\alpha(i)} = \text{diag}(\sqrt{\alpha_1(i)}, \sqrt{\alpha_2(i)}, ..., \sqrt{\alpha_p(i)}) \quad \text{for } i = 1, 2,$$

it can be shown that

$$D^{1/2}_{\alpha(2)} \eta \, D_{\beta(2)} \eta^* \, D^{1/2}_{\alpha(2)} - D^{1/2}_{\alpha(1)} \eta \, D_{\beta(1)} \eta^* \, D^{1/2}_{\alpha(1)}$$

is positive semi-definite and hence $\lambda_{j2} \geqslant \lambda_{j1}$ for all $j = 1, 2, ..., c$. Since g is a monotone increasing function of each λ_j separately, we have

$$h_2 \geqslant h_1 \qquad \text{and} \qquad h_2 \leqslant d \Rightarrow h_1 \leqslant d.$$

This proves $P(h_1 \leqslant d) \geqslant P(h_2 \leqslant d)$ for all $d \geqslant 0$. This proves (3.1).

4. Invariant Test Procedures for Location Parameters

4.a. *Real Variables*

In the notations (2.4), we have

$$(Y - B\xi A)(V'V)^- (Y - B\xi A)' = S + (W - B\xi F)(W - B\xi F)',$$

where

$$S = YA_1'(A_1 V'VA_1')^- A_1 Y' > 0,$$

$$FF' = (AV_0^- A')(AV_0^- V'VV_0^- A')^+ (AV_0^- A'),$$

and $W = YV_0^- A'(AV_0^- A')^{-1} F$ is an $n \times t$ matrix.

We shall take the joint density of S and W as

$$c \, |\theta\theta'|^{-s/2} \, |S|^{1/2(s-t-p-1)} f[\theta^{-1}(S + (W - B\xi F)(W - B\xi F)')\theta'^{-1}] \tag{4.1}$$

for all $S > 0$ and $W \in \mathcal{R}^{p \times t}$, where $c = \Pi^{(1/2)p(s-t)} / \Gamma_p(\frac{1}{2}(s-t))$ and

$$\Gamma_p(n) = \Pi^{(1/4)p(p-1)} \prod_{j=1}^{p} \Gamma\left(n - \frac{j-1}{2}\right).$$

Let B_1 be a $p \times (p-q)$ matrix such that $B_1'B=0$ and $B_1'B_1 = I_{p-q}$. Under normality of the random variables η, the likelihood ratio test procedure for testing $H_0(\xi F=0)$ is to reject H_0 if

$$\lambda = |\hat{\Sigma}|/|S+WW'| \leqslant d, \tag{4.2}$$

where

$$\begin{aligned}\hat{\Sigma} &= S+(I-B(B'S^{-1}B)^{-1}B'S^{-1})WW'(I-S^{-1}B(B'S^{-1}B)^{-1}B') \\ &= S+SB_1(B_1'SB_1)^{-1}B_1'WW'B_1(B_1'SB_1)^{-1}B_1'S\end{aligned}$$

and $|\hat{\Sigma}| = |S|\,|B_1'(S+WW')B_1|/|B_1'SB_1|$ (see, for example, Khatri [6]).

To show this test is null-robust under (4.1), let $B_0=(B, B_1)$ be a nonsingular matrix and $(B_0'B\xi F)' = (\xi_1', 0)$ with $\xi_1 = B'B\xi F$. Use the transformations

$$S_1 = B_0'SB_0 \qquad \text{and} \qquad W_1 = B_0'W.$$

Then, the joint density of S_1 and W_1 is obtained from (4.1) by replacing $B\xi F$ by $B_0'B\xi F$, θ by $\theta_1 = B_0'\theta$, S by S_1 and W by W_1. Then, testing H_0 $(\xi_1=0)$ against H $(\xi_1 \neq 0)$ is invariant under the transformation

$$S_1 \to PS_1P' \qquad \text{and} \qquad W_1 \to PW_1\Delta, \tag{4.3}$$

where

$$P = \begin{pmatrix} P_{11} & P_{12} \\ 0 & P_{22} \end{pmatrix},$$

P_{11}, and P_{22} are nonsingular $p\times p$, $q\times q$, and $(p-q)\times(p-q)$ matrices and $\Delta \in O(t)$. Hence, write $S_2 = S_1 + W_1W_1' = TT'$,

$$U = T^{-1}W_1 = \begin{array}{c} \begin{pmatrix} U_1 \\ U_2 \end{pmatrix} \begin{matrix} q \\ p-q \end{matrix} \\ t \end{array}, \qquad T = \begin{array}{c} \begin{pmatrix} T_{11} & T_{12} \\ 0 & T_{22} \end{pmatrix} \begin{matrix} q \\ p-q \end{matrix} \\ \begin{matrix} q & \quad p-q \end{matrix} \end{array}$$

and define

$$S_3 = \begin{pmatrix} T_{11} & T_{12}-\xi_1U_2' \\ 0 & T_{22} \end{pmatrix} \begin{pmatrix} T_{11}' & 0 \\ T_{12}'-U_2\xi_1' & T_{22}' \end{pmatrix}$$

and

$$h(U, S_3; \xi) = \begin{pmatrix} \xi_1(I-U_2'U_2)\xi_1' - T_{11}U_1\xi_1' - \xi_1U_1'T_{11}' & 0 \\ 0 & 0 \end{pmatrix}.$$

Then, the joint density of S_3 and U is

$$c\,|\theta_1\theta_1'|^{-s/2}\,|S_3|^{(1/2)(s-p-1)}\,|I_p-UU'|^{(1/2)(s-t-p-1)} \times f[\theta_1^{-1}(S_3+h(U,S_3;\xi))\,\theta_1'^{-1}] \tag{4.4}$$

for all $S_3>0$ and $U\in\mathscr{R}^{p\times t}$ such that $I_p-UU'>0$. Note that $h(U,S_3;\xi_1)=0$ when $\xi_1=0$, and $\lambda=|I_p-UU'|/|I_{p-q}-U_2U_2'|$. Further, under H_0, the distribution of U does not depend on the structure of f and hence the tests based on U are null robust. Consequently, the likelihood ratio test is null robust. We can write the other test statistics as

$$\operatorname{tr}Q_1,\quad \operatorname{tr}Q_1(I_t+Q)^{-1},\quad \mathrm{Ch}_{\max}Q_1,\quad \mathrm{Ch}_{\max}Q_1(I_t+Q)^{-1},\qquad \text{etc.,} \tag{4.5}$$

where $Q=W'S^{-1}W$, $Q_1=W'S^{-1}B(B'S^{-1}B)^{-1}B'S^{-1}W$, $I_t-U_2'U_2=(I_t+Q-Q_1)^{-1}$, $I_t-U'U=(I_t+Q)^{-1}$, and $U_1'U_1=(I_t+Q-Q_1)^{-1}-(I_t+Q)^{-1}$. Since the above statistics are functions of U, they are null robust. Let the null distribution of U be

$$g_0(U)=\{\Gamma_p(\tfrac{1}{2}s)/\Pi^{(1/2)pt}\Gamma_p((s-t)/2)\}\,|I_p-UU'|^{(1/2)(s-t-p-1)} \tag{4.6}$$

for all $U\in\mathscr{R}^{p\times t}$ such that $I_p-UU'>0$.

To show that the tests based on U are unbiased, we shall assume that f is spherical and convex; that is,

$$f(P)=f(\Gamma P\Gamma')\qquad \text{for all}\quad \Gamma\in O(P)$$

and $f(\alpha P_1+(1-\alpha)P_2)\leqslant\alpha f(P_1)+(1-\alpha)f(P_2)$ for all $P_1,\ P_2\in\mathscr{R}^{p\times p}$, $\alpha\in[0,1]$. Let us write

$$\theta_1^{-1}=\Gamma_1\begin{pmatrix}G_{11} & G_{12}\\ 0 & G_{22}\end{pmatrix},\qquad \xi_1=\Gamma_2D_\delta\Delta,$$

$$\Gamma=\Gamma_1\begin{pmatrix}\Gamma_2 & 0\\ 0 & I_{p-q}\end{pmatrix},\qquad U\Delta'=U_0,$$

$$\begin{pmatrix}\Gamma_2' & 0\\ 0 & I_{p-q}\end{pmatrix}\begin{pmatrix}G_{11} & G_{12}\\ 0 & G_{22}\end{pmatrix}S_3\begin{pmatrix}G_{11}' & 0\\ G_{12}' & G_{22}'\end{pmatrix}\begin{pmatrix}\Gamma_2 & 0\\ 0 & I_{p-q}\end{pmatrix}$$
$$=S_4=XX'\qquad\text{with}\quad X=\begin{pmatrix}X_{11} & X_{12}\\ 0 & X_{22}\end{pmatrix},$$

where $\Gamma_1\in O(p)$, $\Gamma_2\in O(q)$, $\Delta\in O(t)$, and $\Gamma\in O(p)$. Then, the joint density of S_4 and U_0 is given by

$$c\,|S_4|^{(1/2)(s-p-1)}\,|I_p-U_0U_0'|^{(1/2)(s-t-p-1)}f(S_4+h_1(U_0,S_4;D_\delta)) \tag{4.7}$$

for all $S_4 > 0$ and $U_0 \in \mathscr{R}^{p \times t}$ such that $I_p - U_0 U_0' > 0$, where the sphericity condition of f is used and

$$h_1(U_0, S_4; D_\delta) = \begin{pmatrix} -X_{11} U_{01} D_\delta' - D_\delta U_{01}' X_{11}' + D_\delta (I_t - U_{02}' U_{02}) D_\delta' & 0 \\ 0 & 0 \end{pmatrix}$$

and

$$U_0 = \underset{t}{\begin{pmatrix} U_{01} \\ U_{02} \end{pmatrix}} \begin{matrix} q \\ p-q \end{matrix}.$$

Thus, the power of the test based on $U_0 U_0' = UU'$ depends on the eigenvalues of $(\xi_1 \xi_1')$.

For integrating over S_4, we write the density of X_{11}, X_{12}, X_{22}, and U_0 as

$$g_0(U_0)\, c_0\, |X_{11}|_+^{s-p}\, |X_{22}|_+^{s-p+q}$$

$$f\begin{pmatrix} (X_{11} - D_\delta U_{01}')(X_{11} - D_\delta U_{01}')' + X_{12} X_{12}' + D_\delta (I_t - U_0' U_0) D_\delta' & X_{12} X_{22}' \\ X_{22} X_{12}' & X_{22} X_{22}' \end{pmatrix}$$

for all $X_{11} \in \mathscr{R}^{q \times q}$, $X_{22} \in \mathscr{R}^{(p-q)(p-q)}$, $X_{12} \in \mathscr{R}^{q \times (p-q)}$, and $U_0 \in \mathscr{R}^{p \times t}$ such that $I - U_0 U_0' > 0$, where $g_0(U_0)$ is defined in (4.6) and

$$c_0 = \Pi^{(1/2)ps} \Gamma_{p-q}(\tfrac{1}{2}(p-q))\, \Gamma_q(q/2) / \Pi^{(1/2)(p-q)^2 + (1/2)q^2} \Gamma_p(s/2).$$

Integrating over X, we get the noncentral density of U_0, denoted by $g_H(U_0)$. Let $g_H(U_0)/g_0(U_0) = g_n(U_0)$. Notice that $g_n(U_0) = g_n(\Delta_1 U_0)$ for all $\Delta_1 \in O(p)$. Hence, $g_n(U_0) = g_n(-U_0)$. Hence, by convexity of f and for any $a \in [0, 1]$,

$$g_n(U_0) = g_n(-U_0) = a g_n(-U_0) + (1-a)\, g_n(U_0) \geqslant g_n((1-2a)\, U_0) \tag{4.8}$$

and $g_n(U_0)$ depends only on $U_0' U_0$. Since $(1-2a)^2 < 1$, it follows that $g_n(U_0)$ is isotonic. This shows

THEOREM 4.1. *If the density of η is $f(\eta\eta')$, then the invariant tests based on the elements of U are null robust. Further, if f is spherical and convex, then the tests based on $U_0' U_0$ are unbiased. For the special case, $t = 1$ and $p = q$, the test is UMPI if f is convex and spherical.*

4.b. *Complex Variables*

We can proceed as in the real case and, without actual derivations, the final result is given by

THEOREM 4.2. *For the complex variables in model* (1.1), *the test based on the elements of U is null robust and its null density function is given by*

$$g_0(U) = \{\tilde{\Gamma}_p(s)/\Pi^{pt}\tilde{\Gamma}_p(s-t)\} \, |I_p - UU^*|^{s-t-p-1} \tag{4.9}$$

for all $U \in \mathscr{C}^{p\times t}$ (a space of $p \times t$ matrices defined on the complex field $\mathscr{C}$) such that $I_p - UU^*$ is positive definite where $\tilde{\Gamma}_p(s) = \Pi^{(1/2)p(p-1)} \prod_{\delta=1}^{p} \Gamma(s-j+1)$. If f is convex and spherical, then the tests based on the U^*U are unbiased. For the special case $t = 1$ and $p = q$, this test will become UMPI if f is convex and spherical.

As in the real case, we can propose various statistics for testing $H_0(\xi = 0)$ against $H(\xi \neq 0)$. Note that

$$U^*U = W^*(S + WW^*)^{-1} W, \qquad I_t - U_2^*U_2 = (I_t + Q - Q_1)^{-1},$$

and

$$U_1^*U_1 = (I + Q - Q_1)^{-1} - (I + Q)^{-1},$$

where $Q = F^*Z^*S^{-1}ZF$, $Q_1 = F^*Z^*S^{-1}B(B^*S^{-1}B)^{-1} B^*S^{-1}ZF$, and $Q = Q_1 + Q_2$. The usual test statistics are

$$T_0 = \operatorname{tr}(I_t + Q)^{-1} Q_1, \qquad \Lambda = |I_t + Q - Q_1|/|I + Q|,$$

$$\mathrm{Ch}_{\max}(Q_1(I + Q)^{-1}), \qquad T_{01} = \operatorname{tr}((I_t + Q_2)^{-1} Q_1),$$

$$T_{02} = \operatorname{tr} Q_1, \qquad \Lambda_1 = |I_t + Q_1|^{-1}, \qquad \mathrm{Ch}_{\max}(Q_1), \qquad \text{etc.}$$

Their distributions can be obtained from (4.9), but this is omitted at this stage.

Remark. Suppose, we are interested in testing $H(C\xi D = 0)$ against $H(C\xi D \neq 0)$. Let $C_0 = \binom{C}{C_1}$ and $D_0 = (D, D_1)$ be nonsingular matrices. Then

$$B\,\xi A = BC_0^{-1}(C_0 \xi\, D_0)\, D_0^{-1} A = B_1\, \delta A_1$$

where

$$C_0\, \xi D_0 = \delta = \begin{pmatrix} \delta_{11} & \delta_{12} \\ \delta_{21} & \delta_{22} \end{pmatrix}.$$

Then $H_0(C\xi D = 0) \Leftrightarrow H_0(\delta_{11} = 0)$. Thus, if a test procedure for testing $H_0(\delta_{11} = 0)$ against $H(\delta_{11} \neq 0)$ is obtained, then we can get the test produce for testing $C\xi D$. This is under investigation; see also Kariya [4] and Kariya and Sinha [5]. The results corresponding to $H_0(\delta_{11} = 0)$ have been obtained by the author and will appear at the Proceedings of the Second International Tampere Conference in Statistics (1987) held at Tampere, Finland.

5. Structure on Σ

5.1. *Test for Sphericity (Real Variables)*

The sphericity hypothesis is $H_0(\Sigma = \theta\theta' = \sigma^2 G)$, where G is a known $p \times p$ positive definite matrix and σ^2 is unknown. For the normal error variables, Khatri [7] gave the likelihood ratio test procedure as

$$\text{Reject } H_0 \text{ if } \lambda = |S|\,|I_t + W'G_2W|/|G| \\ \times \{(\operatorname{tr} G^{-1}S + \operatorname{tr} W'G_1W)/p\}^p \leqslant d,$$

where

$$G_1 = G^{-1} - G^{-1}B(B'G^{-1}B)^{-1}B'G^{-1} = B_1(B_1'GB_1)^{-1}B_1',$$
$$G_2 = B_1(B'SB_1)^{-1}B' = S^{-1} - S^{-1}B(B'S^{-1}B)^{-1}B'S^{-1}.$$

and the joint density function of S and W is given by

$$c\,|\Sigma|^{-s/2}\,|S|^{(1/2)(s-t-p-1)} \\ g[\operatorname{tr} \Sigma^{-1}\{S + (W - B\xi F)(W - B\xi F)'\}] \tag{5.1}$$

for $S > 0$ and $W \in \mathscr{R}^{p \times t}$, where c is defined in (4.1). Let $B_0 = (G^{-1/2}B(B'G^{-1}B)^{-1/2}, G^{1/2}B_1(B_1'GB_1)^{-1/2})$ be an orthogonal matrix and

$$S_1 = B_0'G^{-1/2}SG^{-1/2}B_0 = \begin{pmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{pmatrix}, \quad W_1 = B_0'G^{-1/2}W = \begin{pmatrix} W_{11} \\ W_{12} \end{pmatrix}. \tag{5.2}$$

Then the joint density of S_1 and W_1 is

$$c\,|\Sigma_1|^{-s/2}\,|S_1|^{(1/2)(s-t-p-1)} \\ \times g[\operatorname{tr} \Sigma_1^{-1}(S + (W_1 - B_0'G^{-1/2}B\xi F)(W_1 - B_0'G^{-1/2}B\xi F)')], \tag{5.3}$$

where $\Sigma_1 = B_0'G^{-1/2}\Sigma G^{-1/2}B_0$ and $B'G^{-1/2}B_0 = ((B'G^{-1}B)^{1/2}, 0)$. Further

$$\lambda = |S_1|\,|S_{22} + W_{12}W_{12}'|/\{(\operatorname{tr} S_1 + \operatorname{tr} W_{12}'W_{12})/p\}^p\,|S_{22}|.$$

Under H_0 $(\Sigma = \sigma^2 G)$, $\Sigma_1 = \sigma^2 I_p$. Let $y = \operatorname{tr} S_1 + \operatorname{tr} W_{12}W_{12}'$, $X_2 = W_{12}/\sqrt{y}$, and $M = S_1/y$. Then, the Jacobian of the transformation is

$$J(W_{12}, S_1 \to X_2, M, y) = y^{(1/2)p(p+1) + (1/2)(p-q)t - 1}.$$

Hence, under H_0, the joint density of $X_1 = W_{11} - (B'G^{-1}B)^{1/2}\,\xi F$, y, X_2, and M is

$$c\sigma^{-sp}\,|M|^{(1/2)(s-t-p-1)}\,y^{(1/2)(ps-qt)-1}g((y + \operatorname{tr} X_1X_1')/\sigma^2) \tag{5.4}$$

for all $M>0$, $X_2 \in \mathscr{R}^{(p-q)xt}$ such that $\operatorname{tr} M + \operatorname{tr} X_2 X_2' = 1$ and $X \in \mathscr{R}^{q \times t}$, $\gamma \in \mathscr{R}$. Notice that

$$\lambda = p^p \frac{|M| \, |M_{22} + X_2 X_2'|}{|M_{22}|}$$

and the joint distribution of M and X_2 is given by

$$c_0 \, |M|^{(1/2)(s-t-p-1)} \qquad \text{for} \quad \operatorname{tr} M + \operatorname{tr} X_2 X_2' = 1, \tag{5.5}$$

where

$$c_0 = \Gamma\left(\frac{ps-qt}{2}\right) \Big/ \Pi^{(1/2)t(p-q)} \, \Gamma_p\left(\frac{s-t}{2}\right).$$

Thus, the tests based on M and X_2 must be null robust and hence the likelihood ratio test is null robust. Notice that

$$E\lambda^h = \frac{p^{hp} \Gamma\left(\frac{ps-qt}{2}\right) \Gamma_{p-q}\left(\frac{s-t}{2}\right)}{\Gamma_p\left(\frac{s-t}{2}\right) \Gamma_{p-q}\left(\frac{s}{2}\right)}$$

$$\times \int_{\mathscr{D}} |M_1|^{(1/2)(s-t+2h-p-1)} \, |M_2|^{(1/2)(s-p-q+2h-1)} \, dM_1 \, dM_2 \, dX,$$

where $\mathscr{D}$: $\{M_1, M_2, X\}$, $\operatorname{tr} M_1 + \operatorname{tr} M_2 + \operatorname{tr} XX' = 1$, M_1 and M_2 are $q \times q$, and $(p-q) \times (p-q)$ positive definite matrices and $X \in \mathscr{R}^{q \times (p-q)}\}$. This gives

$$E\lambda^h = \frac{p^{hp} \Gamma\left(\frac{ps-qt}{2}\right) \Gamma_q\left(\frac{s-t-p+q}{2} + h\right) \Gamma_{p-q}\left(\frac{s}{2} + h\right)}{\Gamma_q\left(\frac{(s-t-p+q)}{2}\right) \Gamma_{p-q}\left(\frac{s}{2}\right) \Gamma\left(\frac{ps-qt}{2} + ph\right)}. \tag{5.6}$$

Notice that this result does not agree exactly with the result mentioned by Khatri [7, p. 113] after making the changes $s \to n$, $p \to p$, $t \to p$, and $q \to r$. Hence, the result of Khatri (7, p. 113) should be corrected as mentioned above.

5.2. *Test for Sphericity (Complex Variables)*

The likelihood ratio test for $H_0(\theta\theta^* = \sigma^2 G)$ under the complex multivariate normal distribution for the error variables of model (1.1) is to reject H_0 if

$$\lambda = |S| \, |I_t + W^* G_2 W| / |G| \, \{(\operatorname{tr} G^{-1} S + \operatorname{tr} W^* G_1 W)/p\}^p \leqslant d,$$

where

$$G_1 = G^{-1} - G^{-1}B(B^*G^{-1}B)^{-1} B^*G^{-1} = B_1(B_1^*GB_1)^{-1} B_1^*,$$
$$G_2 = S^{-1} - S^{-1}B(B^*S^{-1}B)^{-1} B^*S^{-1} = B_1(B_1^*SB_1)^{-1}B_1^*$$

and the joint density of S and W is given by

$$c\,|\Sigma|^{-s}\,|S|^{(s-t-p)} \times g[\operatorname{tr} \Sigma^{-1}(S + (W - B\xi F)(W - B\xi F)^*)] \tag{5.7}$$

for S being Hermitian positive definite and $W \in \mathscr{C}^{p \times t}$ with $c = \Pi^{p(s-t)}/\tilde{\Gamma}_p(s-t)$. Now, proceeding exactly in the real case, it can be shown that under H_0, $\{M, X_2\}$ and $\{y, X_1\}$ are independently distributed, by taking $B_0 = (G^{-1/2}B(B^*G^{-1}B)^{-1/2}, G_1^{1/2}B_1(B_1^*GB_1)^{-1/2})$, a unitary matrix,

$$M = B_0^* G^{-1/2} S G^{-1/2} B_0 / y,$$
$$X_2 = (B_1^*GB_1)^{-1/2} B_1^* W/\sqrt{y},$$
$$y = \operatorname{tr} G^{-1}S + \operatorname{tr} W^*G_1 W,$$
$$X_1 = (B^*G^{-1}B)^{-1/2} B^*G^{-1}W - (B^*G^{-1}B)^{1/2}\xi F,$$

and $\lambda = p^p\,|M|\,|M_{22} + X_2X_2^*|/|M_{22}|$, and the joint density of M and X_2 under H_0 is

$$c_0\,|M|^{s-t-p} \qquad \text{for} \quad \operatorname{tr} M + \operatorname{tr} X_2X_2^* = 1, \tag{5.8}$$

where $c_0 = \Gamma(ps-qt)/\Pi^{t(p-q)}\,\tilde{\Gamma}_p(s-t)$. Thus, the test based on M and X_2 are null robust and therefore the likelihood ratio test is null robust. We observe that similar to (5.6), we have

$$E\lambda^h = \frac{p^{ph}\Gamma(ps-qt)\,\tilde{\Gamma}_q(s-t-p+q+h)\,\tilde{\Gamma}_{p-q}(s+h)}{\tilde{\Gamma}_q(s-t-p+q)\,\tilde{\Gamma}_{p-q}(s)\,\Gamma(ps-qt+ph)}.$$

5.2a. *Test for Intraclass correlation Structure for Σ (Real Variables)*

For the normal distribution for the error variables, Khatri [7] considered the likelihood ratio test procedure for testing H_0 $(\Sigma = \sigma_1 G + \sigma_2 \mathbf{w}\mathbf{w}')$, where G and $\mathbf{w}$ are known and $\mathbf{w} = B\boldsymbol{\delta}$ for some $\boldsymbol{\delta}$, against $H(\Sigma \neq \sigma_1 G + \sigma_2 \mathbf{w}\mathbf{w}')$. This test procedure is to reject H_0 if

$$\lambda = \frac{|S|\,|I_t + W'G_2W|\,(\mathbf{w}'G^{-1}\mathbf{w})(p-1)^{p-1}}{\left(\begin{matrix}|G|\,(\mathbf{w}'G^{-1}SG^{-1}\mathbf{w})[\operatorname{tr} G^{-1}S + \operatorname{tr} W'G_1W \\ -\mathbf{w}'G^{-1}SG^{-1}\mathbf{w}/\mathbf{w}'G^{-1}\mathbf{w}]^{p-1}\end{matrix}\right)} \leqslant d, \tag{5.9}$$

where G_1, G_2, S, and W are the same as defined in Section (5.1) and the joint density of S and W is given by (5.1). Let B_2 be $q \times q$ orthogonal matrix such that its first column vector is

$$(B'G^{-1}B)^{-1/2}\, B'G^{-1}\mathbf{w}/(\mathbf{w}'\, G^{-1}B(B'G^{-1}B)^{-1}\, B'G^{-1}\mathbf{w})^{1/2}$$

and $B_3 = \operatorname{diag}(B_2, I_{p-q})$. Then, in the notations of Section 5.1, let

$$B_3'B_0'G^{-1/2}SG^{-1/2}B_0B_3 = S_1, \qquad B_3'B_0'G^{-1/2}W = W_1,$$

and

$$\Sigma_1 = B_3'B_0'G^{-1/2}\Sigma G^{-1/2}B_0B_3.$$

It is easy to see that the density of S_1 and W_1 is the same as (5.3) after replacing $B_0'G^{-1/2}B\xi F$ by $(B_3'B_0'G^{-1/2}B\xi F)$. Observe that under H_0,

$$\Sigma_1 = \operatorname{diag}(\sigma_1 + b\sigma_2, \sigma_1 I_{p-1})$$

with

$$b = \mathbf{w}'\, G^{-1}B(B'G^{-1}B)^{-1}\, B'G^{-1}\mathbf{w} = \mathbf{w}'\, G^{-1}\mathbf{w}$$

on account of $\mathbf{w} = B\boldsymbol{\delta}$. Let us partition S_1 and W_1 as

$$S_1 = \begin{array}{c} \\ 1 \\ p-1 \\ \\ \end{array}\!\!\begin{array}{cc} & \\ \multicolumn{2}{c}{\begin{pmatrix} s_{11} & \mathbf{s}' \\ \mathbf{s} & S_{11} \end{pmatrix}} \\ 1 & p-1 \end{array}, \qquad W_1 = \begin{array}{c}\begin{pmatrix} W_{11} \\ W_{12} \end{pmatrix} \\ t \end{array}\!\!\begin{array}{c} q \\ p-q \\ \\ \end{array}, \qquad \frac{1}{\sqrt{s_{11}}}\, S_{11}^{-1/2}\mathbf{s} = \mathbf{z}.$$

Then,

$$\begin{aligned} s_{11} &= \mathbf{w}'\, G^{-1}B(B'G^{-1}B)^{-1}\, B'G^{-1}SG^{-1}B(B'G^{-1}B)^{-1}\, B'G^{-1}\mathbf{w}/b \\ &= \mathbf{w}'\, G^{-1}SG^{-1}\mathbf{w}/b \end{aligned}$$

and

$$\lambda = \frac{|S_{11}|\,(1-\mathbf{z}'\mathbf{z})\,|I_t + W_{12}'S_{22}^{-1}W_{12}|\,(p-1)^{p-1}}{(\operatorname{tr} S_{11} + \operatorname{tr} W_{12}'W_{12})^{p-1}} = (1-\mathbf{z}'\mathbf{z})\lambda_1, \tag{5.10}$$

where λ_1 is the likelihood ratio test statistic based on S_{11} and W_{12} (i.e., $q \to q-1$ and $p \to p-1$ are changed in (5.1a). Under H_0, we observe that $\mathbf{z}$ and λ_1 are independently distributed, and the density of $\mathbf{z}'\mathbf{z}$ is beta $((p-1)/2,\ (s-t-p+1)/2)$. We observe that the tests based on $\mathbf{z}'\mathbf{z}$, $M_1 = S_{11}/y$, and $X_2 = W_{12}/\sqrt{y}$ with $y = \operatorname{tr} S_{11} + \operatorname{tr} W_{12}'W_{12}$ are null robust and therefore the likelihood ratio test is null robust.

Using (5.6) after changing $p \to p-1$ and $q \to q-1$ and using the density of $\mathbf{z}'\mathbf{z}$, we see that

$$E\lambda^h = \frac{\begin{pmatrix} (p-1)^{(p-1)h}\,\Gamma_q((s-t-p+q)/2+h) \\ \times\,\Gamma_{p-q}((s/2)+h)\,\Gamma((ps-qt-s+t)/2)\,\Gamma((s-t)/2) \end{pmatrix}}{\begin{pmatrix} \Gamma_q((s-t-p+q)/2)\,\Gamma_{p-q}(s/2) \\ \times\,\Gamma((ps-qt-s+t)/2)+ph-h)\,\Gamma((s-t)/2+h) \end{pmatrix}}. \qquad (5.11)$$

Notice that this result does not agree exactly with that of Khatri [7, p. 114] after making the changes $s \to n$, $p \to p$, $t \to t$, and $q \to r$. Hence, the result of Khatri [7, p. 114] should be corrected as mentioned above.

5.2b. *Test for Intraclass Correlation Structure for Σ (Complex Variables)*

Here, $\Sigma = \theta\theta^*$ and under H_0, $\Sigma = \sigma_1 G + \sigma_2 \mathbf{w}\mathbf{w}^*$, where $\mathbf{w} = B\,\boldsymbol{\delta}$ for some $\boldsymbol{\delta}$. Under the complex multivariate normal distribution for the error variables of model (1.1), the likelihood ratio test procedure for H_0 is to reject H_0 if

$$\lambda = \frac{|S|\,|I_t + W^*G_2W|\,(\mathbf{w}^*G^{-1}\mathbf{w})(p-1)^{p-1}}{\begin{pmatrix} |G|\,(\mathbf{w}^*\,G^{-1}SG^{-1}\mathbf{w})[\operatorname{tr} G^{-1}S + \operatorname{tr} W^*G_1W \\ -\,\mathbf{w}^*G^{-1}SG^{-1}\mathbf{w}/(\mathbf{w}^*\,G^{-1}\mathbf{w})]^{p-1} \end{pmatrix}} \leqslant d,$$

where G_2, G_1, S, and W are the same as defined in Section 5.1 and the joint density of S and W is the same as given in (5.7).
Now, we can proceed in the same way as in the real variables case and define B_2 as a unitary $q \times q$ matrix such that its first column is $(B^*G^{-1}B)^{-1/2}\,B^*G^{-1}\mathbf{w}/\sqrt{b}$, $b = \mathbf{w}^*G^{-1}B(B^*G^{-1}B)^{-1}\,B^*G^{-1}\mathbf{w} = \mathbf{w}^*G^{-1}\mathbf{w}$, and $B_3 = \operatorname{diag}(B_2, I_{p-q})$. Define

$$B_3^*B_0^*G^{-1/2}SG^{-1/2}B_0B_3 = S_1 = \begin{pmatrix} s_{11} & \mathbf{s}^* \\ \mathbf{s} & S_{11} \end{pmatrix},$$

$$B_3^*B_0^*G^{-1/2}W = W_1 = \begin{array}{c} \begin{pmatrix} W_{12} \\ W_{22} \end{pmatrix} \end{array}\!\!\begin{array}{l} q \\ p-q \end{array},$$

$$\mathbf{z}^*\mathbf{z} = \mathbf{s}^*S_{11}^{-1}\mathbf{s}/s_{11}, \qquad y = \operatorname{tr} S_{11} + \operatorname{tr} W_{12}^*W_{12},$$

$$M_1 = S_{11}/y, \qquad \text{and} \qquad X_2 = W_{22}/\sqrt{y}.$$

Then, as in Section 5.2a, it can be shown that the tests based on $\mathbf{z}^*\mathbf{z}$, M_1, and X_2 are null robust and, therefore, the likelihood ratio test is null robust. Further,

$$\lambda = (1-\mathbf{z}^*\mathbf{z})\lambda_1, \qquad \lambda_1 = (p-1)^{p-1}\,|M_1|\,|M_{22} + X_2X_2^*|/|M_{22}|,$$

where M_{22} is a submatrix of M_1 obtained by taking the last $(p-q)$ rows and the last $(p-q)$ columns, $\mathbf{z}^*\mathbf{z}$ is distributed as beta$(p-1, s-t-p+1)$ and $\mathbf{z}^*\mathbf{z}$ and λ_1 are independently distributed under H_0. Hence, using Section 5.1b with $p \to p-1$ and $q \to q-1$, we get

$$E\lambda^h = \frac{\begin{pmatrix} (p-1)^{(p-1)h}\, \tilde{\Gamma}_q(s-t-p+q+h)\, \tilde{\Gamma}_{p-q}(s+h) \\ \times \Gamma(ps-qt-s+t)\, \Gamma(s-t) \end{pmatrix}}{\tilde{\Gamma}_q(s-t-p+q)\, \tilde{\Gamma}_{p-q}(s)\, \Gamma(ps-qt-s+t+ph-h)\, \Gamma(s-t+h)}.$$

6. Structure on Σ Due to Uncorrelation

6.1a *Real Variables*

In this situation, the joint density of S and W is the same as given in Section 5.1 and define

$$\Sigma = \begin{array}{c} \begin{pmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{pmatrix} \\ \quad p_1 \qquad p_2 \end{array}, \qquad B = \begin{array}{c} \begin{pmatrix} B_{(1)} \\ B_{(2)} \end{pmatrix} \begin{matrix} p_1 \\ p_2 \end{matrix} \\ q \end{array},$$

Rank $B_{(i)} = q_i$, $q_1 + q_2 = q$, and $p_1 + p_2 = p$. Then, the hypothesis of uncorrelation of the two sets is H_0 $(\Sigma_{12} = 0)$ against $H(\Sigma_{12} \neq 0)$. Let

$$I_{p_i} = B_{(i)}(B'_{(i)}B_{(i)})^- B'_{(i)} + B_{i3}B'_{i3}$$

and

$$B_{i1}B'_{i1} = B_{(i)}(B'_{(i)}B_{(i)})^- B_{(i)}.$$

Then (B_{i1}, B_{i3}) is a $p_i \times p_i$ orthogonal matrix for $i = 1, 2$. Let

$$B_{(\cdot)} = \begin{pmatrix} B_{11} & 0 \\ 0 & B_{21} \end{pmatrix}, \qquad B_1 = \begin{pmatrix} B_{13} & 0 \\ 0 & B_{23} \end{pmatrix}, \qquad \text{and} \qquad B_0 = (B_{(\cdot)}, B_1).$$

Then, $B_0 \in O(p)$. For the normal variables η, Khatri [7] derived the likelihood ratio test statistics $\lambda = \lambda_1 \lambda_2$ where taking

$$S = \begin{array}{c} \begin{pmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{pmatrix} \begin{matrix} p_1 \\ p_2 \end{matrix} \\ p_1 \qquad p_2 \end{array} \qquad \text{and} \qquad W = \begin{array}{c} \begin{pmatrix} W_1 \\ W_2 \end{pmatrix} \begin{matrix} p_1 \\ p_2 \end{matrix} \\ t \end{array},$$

$$\lambda_1 = |S|\, |B'_{13}S_{11}B_{13}|\, |B'_{23}S_{23}B_{23}| / |S_{11}|\, |S_{22}|\, |B'_1SB_1| \tag{6.1}$$

and

$$\lambda_2 = |B_1'(S + WW')B_1| / |B_{13}'(S_{11} + W_1 W_1')B_{13}|\ |B_{23}'(S_{22} + W_2 W_2')B_{23}|. \tag{6.2}$$

Let

$$\Sigma_1 = B_0' \Sigma B_0 = \begin{array}{cc} \begin{pmatrix} \Sigma_{(11)} & \Sigma_{(12)} \\ \Sigma_{(21)} & \Sigma_{(22)} \end{pmatrix} & \begin{array}{l} q \\ p-q \end{array} \\ \begin{array}{cc} q & \quad p-q \end{array} & \end{array}, \qquad \beta = \Sigma_{(12)} \Sigma_{(22)}^{-1}$$

and

$$\Sigma_2 = \Sigma_{(11)} - \beta \Sigma_{(21)}. \tag{6.3}$$

Observe that

$$\Sigma_2 (B_{(\cdot)}' \Sigma^{-1} B_{(\cdot)})^{-1} = \begin{array}{cc} \begin{pmatrix} \Sigma_3 & 0 \\ 0 & \Sigma_4 \end{pmatrix} & \begin{array}{l} q_1 \\ q_2 \end{array} \\ \begin{array}{cc} q_1 & \quad q_2 \end{array} & \end{array}$$

under $H_0(\Sigma_{12} = 0)$, and

$$\beta = \begin{array}{cc} \begin{pmatrix} \beta_1 & 0 \\ 0 & \beta_2 \end{pmatrix} & \begin{array}{l} q_1 \\ q_2 \end{array} \\ \begin{array}{cc} p_1 - q_1 & \quad p_2 - q_2 \end{array} & \end{array}$$

under H_0, with $\beta_i = B_{i1}' \Sigma_{ii} B_{i3} (B_{i3}' \Sigma_{ii} B_{i3})^{-1}$ for $i = 1, 2$.

We make the similar transformations:

$$S_1 = B_0' S B_0 = \begin{array}{cc} \begin{pmatrix} S_{(11)} & S_{(12)} \\ S_{(21)} & S_{(22)} \end{pmatrix} & \begin{array}{l} q \\ p-q \end{array} \\ \begin{array}{cc} q & \quad p-q \end{array} & \end{array}$$

$$W_{(\cdot)} = B_0'(W - B\xi F) = \begin{array}{cc} \begin{pmatrix} W_{(1)} \\ W_{(2)} \end{pmatrix} & \begin{array}{l} q \\ p-q \end{array} \\ t & \end{array}$$

$$\begin{aligned} S_{(11\cdot 2)} &= S_{(11)} - S_{(12)} S_{(22)}^{-1} S_{(21)} \\ &= \begin{array}{cc} \begin{pmatrix} R_{11} & R_{12} \\ R_{21} & R_{22} \end{pmatrix} & \begin{array}{l} q_1 \\ q_2 \end{array} \\ \begin{array}{cc} q_1 & \quad q_2 \end{array} & \end{array}, \end{aligned}$$

$$Z = S_{(12)},$$

where $S_{(22)} = TT'$ and

$$T = \begin{pmatrix} T_{11} & 0 \\ T_{21} & T_{22} \end{pmatrix}$$

with T_{ii} being a $(p_i - q_i) \times (p_i - q_i)$ matrix.

Notice that $W_{(1)}$ depends on ξ, while $W_{(2)}$ does not depend on ξ, and hence $W_{(1)}$ will not be useful in the test statistic. Let

$$X_1 = \Sigma^{-1/2}(W_{(1)} - BW_{(2)}).$$

Then, the joint density of X_1, Z, $S_{(22)}$, $S_{(11.2)}$, and $W_{(2)}$ is given by

$$\begin{aligned} c\,|\Sigma_2|^{-(s-t)/2}\,&|\Sigma_{(22)}|^{-s/2}\,|S_{(11.2)}|^{(1/2)(s-t-p-1)} \\ &\times |S_{(22)}|^{(1/2)(s-t-p+q-1)} g[\operatorname{tr} X_1 X_1' \\ &+ \operatorname{tr} \Sigma_2^{-1} S_{(11.2)} + \operatorname{tr} \Sigma_2^{-1}(Z - \beta T)(Z - \beta T)' \\ &+ \operatorname{tr} \Sigma_{(22)}^{-1}(S_{22} + W_{(2)} W_{(2)}')] \end{aligned} \tag{6.4}$$

for $X_1 \in \mathscr{R}^{q \times t}$, $S_{(11.2)} > 0$, $Z \in \mathscr{R}^{q \times (p-q)}$, $W_{(2)} \in \mathscr{R}^{(p-q)} \times t$, and $S_{22} > 0$. Under $H_0(\Sigma_{12} = 0)$, we observe that

$$\operatorname{tr} \Sigma_2^{-1} S_{(11.2)} = \operatorname{tr} \Sigma_3^{-1} R_{11} + \operatorname{tr} \Sigma_4^{-1} R_{22}$$

and

$$\begin{aligned} \operatorname{tr} &\Sigma_2^{-1}(Z - \beta T)(Z - \beta T)' \\ &= \operatorname{tr} \Sigma_3^{-1}(Z_{11} - \beta_1 T_{11})(Z_{11} - \beta_1 T_{11})' + \operatorname{tr} \Sigma_3^{-1} Z_{12} Z_{12}' \\ &\quad + \operatorname{tr} \Sigma_4^{-1}(Z_2 - \beta_2(T_{21}, T_{22}))(Z_2 - \beta(T_{21}, T_{22}))', \end{aligned}$$

where

$$Z = \begin{pmatrix} Z_{11} & \vdots & Z_{12} \\ \cdots & \cdots & \cdots \\ & Z_2 & \end{pmatrix}, \ Z_{11}, Z_{12}, \text{and } Z_2$$

are respectively $q \times (p-q)$, $q_1 \times (p_1 - q_1)$, $q_1 \times (p_2 - q_2)$, and $q_2 \times (p-q)$. Define Γ by a $(p-q) \times (p-q)$ orthogonal matrix defined by

$$\Gamma = (\Gamma_1, \Gamma_2), \qquad \Gamma_2 = \begin{pmatrix} T_{21}' \\ T_{22}' \end{pmatrix} (T_{21} T_{21}' + T_{22} T_{22}')^{-1/2}$$

$$T_2 = (T_{21} T_{21}' + T_{22} T_{22}')^{1/2},$$

and use the transformation $Z_2\Gamma = (Z_{21}, Z_{22})$. Then

$$\begin{aligned}(Z_2 - \beta_2(T_{21}, T_{22}))(Z_2 - \beta_2(T_{21}, T_{22}))' \\ = (Z_{21}, Z_{22} - \beta_2 T_2)(Z_{21}, Z_{22} - \beta_2 T_2)'.\end{aligned}$$

Let $\Sigma_4^{-1/2}(Z_{22} - \beta_2 T_2) = X_3$ and $\Sigma_3^{-1/2}(Z_{11} - \beta_1 T_{11}) = X_2$. Then, under H_0, the joint density of X_1, X_2, X_3, Z_{12}, Z_{21}, R_{11}, R_{22}, R_{12}, S_{22}, and $W_{(2)}$ is given by

$$\begin{aligned}& c\,|\Sigma_3|^{(1/2)(s-t-p_1+q_1)}\,|\Sigma_4|^{-(s-t-p_2+q_2)/2} \\ & \times |\Sigma_{(22)}|^{-s/2} \begin{vmatrix} R_{11} & R_{12} \\ R_{21} & R_{22} \end{vmatrix}^{(1/2)(s-t-p-1)} |S_{(22)}|^{(1/2)(s-t-p+q-1)} \\ & \times g[\operatorname{tr} X_1 X_1' + \operatorname{tr} X_2 X_2' + \operatorname{tr} X_3 X_3' \\ & + \operatorname{tr} \Sigma_3^{-1}(R_{11} + Z_{12} Z_{12}') + \operatorname{tr} \Sigma_4^{-1}(R_{22} + Z_{21} Z_{21}') \\ & + \operatorname{tr} \Sigma_{(22)}^{-1}(S_{(22)} + W_{(2)} W_{(2)}')]. \end{aligned} \tag{6.5}$$

We can define

$$S_{(22)} + W_{(2)} W_{(2)}' = \begin{array}{cc} \begin{pmatrix} R_{33} & R_{34} \\ R_{43} & R_{44} \end{pmatrix} & \begin{array}{l} p_1 - q_1 \\ p_2 - q_2 \end{array} \\ \begin{array}{cc} p_1 - q_1 & p_2 - q_2 \end{array} & \end{array}$$

and under H_0,

$$\Sigma_{(22)} = \begin{array}{cc} \begin{pmatrix} \Sigma_5 & 0 \\ 0 & \Sigma_6 \end{pmatrix} & \begin{array}{l} p_1 - q_1 \\ p_2 - q_2 \end{array}. \\ \begin{array}{cc} p_1 - q_1 & p_2 - q_2 \end{array} & \end{array}$$

Hence, the test must depend on the following statistics:

$$\begin{aligned} & U_1 = R_{11}^{-1/2} R_{12} R_{22}^{-1/2}, \qquad U_2 = (R_{11} + Z_{12} Z_{12}')^{-1/2}\, Z_{12}, \\ & U_3 = (R_{22} + Z_{21} Z_{21}')^{-1/2}\, Z_{21}, \qquad U_4 = R_{33}^{-1/2} R_{34} R_{44}^{-1/2}, \\ & U_5 = (S_{(22)} + W_{(2)} W_{(2)}')^{-1/2}\, W_{(2)}, \end{aligned}$$

and it can be observed that under H_0,

$$U_1, U_2, U_3, U_4, U_5, \{X_1, X_2, X_3, R_{11} + Z_{12} Z_{12}', R_{22} + Z_{21} Z_{21}', R_{33}, R_{44}\}$$

are independently distributed.

Note that $\lambda_2 = |I - U_4 U_4'| = |S_{(22)} + W_{(2)} W_{(2)}'|/|R_{33}| \cdot |R_{44}|$, is a likelihood ratio test statistic for testing $H_0(B_{13}' \Sigma_{12} B_{23} = 0)$ based on s observations. We have

$$\lambda_1 = \begin{vmatrix} R_{11} & R_{12} \\ R_{21} & R_{22} \end{vmatrix} \Big/ |R_{11} + Z_{12} Z_{12}'| \cdot |R_{22} + Z_{21} Z_{21}'|$$

$$= |I - U_1 U_1'| \cdot |U_2 U_2'| \cdot |I - U_3 U_3'|.$$

Observe that $R_{11} + Z_{12} Z_{12}' = (B_{11}' S_{11} B_{11})^{-1}$, $R_{22} + Z_{21} Z_{21}' = (B_{21}' S_{22}^{-1} B_{21})^{-1}$ and

$$S_{(11.2)} = \begin{pmatrix} R_{11} & R_{12} \\ R_{21} & R_{22} \end{pmatrix} = (B(\cdot) S^{-1} B_{(\cdot)})^{-1}.$$

Thus, any test based on the above statistics is null robust.

6.1b. *Complex Variables*

We can proceed exactly in the sae way as in Section 6.1a for testing the similar structure on Σ. The likelihood ratio statistic is similar to (6.1) and (6.2). We can establish the null robustness, but the details will be omitted as the readers can supply the necessary changes.

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